Camera to Robot-body Calibration Using Planar Mirror Reflections

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1 Introduction

The purpose of this technical report is to detail our approach for extrinsic camera calibration. We describe our approach for determining the transformation between the camera frame, $\{C\}$, and a different frame of interest, $\{B\}$. Frame $\{B\}$ is arbitrary and without loss of generality, we will refer to frame $\{B\}$ as the "base frame." Example base frames vary by application, and include (i) the robot-body frame, if the camera is mounted on a robot, (ii) the room or building frame, if the camera is utilized in a surveillance application, or (iii) the rig mount, if the camera is part of a stereo pair. We consider a scenario in which the mirror moves in front of the camera, allowing the camera to record N_c images of the scene; each image contains the reflections of 3 points, whose coordinates in $\{B\}$ are known. We exploit these observations of point reflections to determine the transformation between $\{B\}$ and $\{C\}$, without knowledge of the mirror's placement with respect to the camera, or of the mirror's motion (cf. Algorithm 1). In what follows, we review the algorithm (Section 2), we present our measurement model (Section 3), and we describe the analytical solution (Section 4). Lastly, we discuss a Maximum-Likelihood approach for refining the computed transformation (Section 5), and we discuss the unobservable cases which arise in the system under consideration (Section 6).

2 Algorithm Overview

Algorithm 1 Computing the Camera to Base-Frame Transformation

Input: Set of 3 points tracked in N_c images
Output: Transformation $\{{}^C_B\mathbf{R}, {}^C\mathbf{p}_B\}$ between camera and base frames
  for each image in N_c **do**
    Convert to Three Point Pose Estimation problem (P3P)
    Solve P3P to obtain combined rotation/reflection transformation: $\{\mathbf{A}, \mathbf{b}\}_\mu$
  end for
  for each triplet of solutions $\{\mathbf{A}, \mathbf{b}\}_\mu$, $\{\mathbf{A}, \mathbf{b}\}_{\mu'}$, $\{\mathbf{A}, \mathbf{b}\}_{\mu''}$ **do**
    Compute mirror configurations from (13)
    Compute camera-to-base rotation ${}^C_B\mathbf{R}$ from (54)
    Compute camera-to-base translation ${}^C\mathbf{p}_B$ from (15)
  end for
  Utilize clustering to select the correct solution $\{{}^C_B\mathbf{R}, {}^C\mathbf{p}_B\}$
  Refine the solution using a maximum-likelihood estimator

3 Measurement Model

First, we present the measurement model that describes each of the camera observations. To simplify the presentation, in this section we focus on the case of a single point, observed in a single image. Consider a point $\mathbf{p}$, whose position with respect to frame $\{B\}$, ${}^{B}\mathbf{p}$, is known[1]. We would like to express the point $\mathbf{p}$ in the camera reference frame $\{C\}$. From geometry (cf. Fig. 1) we have two constraint equations:

$$ {}^{C}\mathbf{p}' = {}^{C}\mathbf{p} + 2d_p {}^{C}\mathbf{n} \tag{1} $$

$$ d_p = d - {}^{C}\mathbf{n}^{\mathrm{T}} {}^{C}\mathbf{p} \tag{2} $$

where ${}^{C}\mathbf{p}'$ is the reflection of ${}^{C}\mathbf{p}$, ${}^{C}\mathbf{n}$ is the mirror normal vector expressed in the camera frame, and d is the distance between the mirror and the camera along the mirror normal vector. Note also that

$$ {}^{C}\mathbf{p} = {}^{C}_{B}\mathbf{R}\,{}^{B}\mathbf{p} + {}^{C}\mathbf{p}_B, \tag{3} $$

where ${}^{C}_{B}\mathbf{R}$ is the matrix which rotates vectors between frames $\{B\}$ and $\{C\}$, and ${}^{C}\mathbf{p}_B$ is the origin of $\{B\}$ with respect to $\{C\}$. We substitute (2) and (3) into (1), and rearrange the terms

$$ \begin{aligned} {}^{C}\mathbf{p}' &= {}^{C}\mathbf{p} + 2d_p {}^{C}\mathbf{n} \\ &= \left(\mathbf{I}_3 - 2{}^{C}\mathbf{n}{}^{C}\mathbf{n}^{\mathrm{T}}\right){}^{C}\mathbf{p} + 2d{}^{C}\mathbf{n} \\ &= \left(\mathbf{I}_3 - 2{}^{C}\mathbf{n}{}^{C}\mathbf{n}^{\mathrm{T}}\right)\left({}^{C}_{B}\mathbf{R}\,{}^{B}\mathbf{p} + {}^{C}\mathbf{p}_B\right) + 2d{}^{C}\mathbf{n} \\ \Leftrightarrow \begin{bmatrix} {}^{C}\mathbf{p}' \\ 1 \end{bmatrix} &= \begin{bmatrix} \left(\mathbf{I}_3 - 2{}^{C}\mathbf{n}{}^{C}\mathbf{n}^{\mathrm{T}}\right) & 2d{}^{C}\mathbf{n} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{C}_{B}\mathbf{R} & {}^{C}\mathbf{p}_B \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{p} \\ 1 \end{bmatrix} \end{aligned} \tag{4} $$

The matrix $\mathbf{I}_3$ is the 3×3 identity matrix, and the vector $\mathbf{0}_{3\times 1}$ is the 3×1 zero vector. The reflection of $\mathbf{p}$, is observed by the camera, and this observation is described by the perspective projection model:

$$ \mathbf{z} = \frac{1}{p_3}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \boldsymbol{\eta} = \mathbf{h}({}^{C}\mathbf{p}') + \boldsymbol{\eta}, \quad {}^{C}\mathbf{p}' = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^{\mathrm{T}} \tag{5} $$

where $\boldsymbol{\eta}$ is the measurement noise, assumed to be zero-mean Gaussian with covariance matrix $\sigma_\eta^2\mathbf{I}_2$. Equations (4) and (5) define our measurement model in which we have expressed ${}^{C}\mathbf{p}'$ as a function of the *known* position vector ${}^{B}\mathbf{p}$, the *unknown* transformation (rotation and translation) between the camera and body frame, $\{{}^{C}_{B}\mathbf{R}, {}^{C}\mathbf{p}_B\}$, and the *unknown* configuration of the mirror with respect to the camera $\{{}^{C}\mathbf{n}, d\}$. We point out that, even though the transformation between the mirror and camera frame has six degrees of freedom (6-dof), only 3-dof appear in the measurement equation. These are expressed by the vector $d{}^{C}\mathbf{n}$, which has 2-dof from the mirror normal, ${}^{C}\mathbf{n}$, and 1-dof from the camera-to-mirror distance, d. The remaining 3-dof, which correspond to rotations about ${}^{C}\mathbf{n}$ and to translations of the origin of the mirror frame in the mirror plane, do not affect the measurements, and are unobservable.

4 Analytical Solution

In this section, we describe our analytical solution for determining the transformation between a camera and base frame of reference. We begin by describing how our method relates to the well known Three Point Perspective Pose Estimation Problem (P3P).

4.1 Three Point Perspective Pose Estimation Problem

We now briefly recap the Three Point Perspective Pose Estimation Problem (P3P) and discuss how it relates to our problem. The goal of P3P is to determine the transformation between a camera frame, $\{C\}$, and a base frame, $\{B\}$, given the known coordinates of three noncollinear points in the base frame, and their corresponding perspective projections in the camera frame. That is, given ${}^{B}\mathbf{p}_i$, and $\mathbf{z}_i$ for $i = 1\ldots 3$, defined as:

$$ \begin{bmatrix} {}^{C}\mathbf{p}_i' \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{C}_{B}\mathbf{R} & {}^{C}\mathbf{p}_B \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{p}_i \\ 1 \end{bmatrix} \tag{6} $$

$$ \mathbf{z}_i = \frac{1}{p_3}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad {}^{C}\mathbf{p}_i' = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}_i^{\mathrm{T}} \tag{7} $$

[1]Throughout this paper, ${}^{X}\mathbf{y}$ denotes the expression of a vector $\mathbf{y}$ with respect to frame $\{X\}$, ${}^{X}_{W}\mathbf{R}$ is the rotation matrix rotating vectors from frame $\{W\}$ to frame $\{X\}$, and ${}^{X}\mathbf{p}_W$ is the position of the origin of frame $\{W\}$, expressed with respect to frame $\{X\}$.

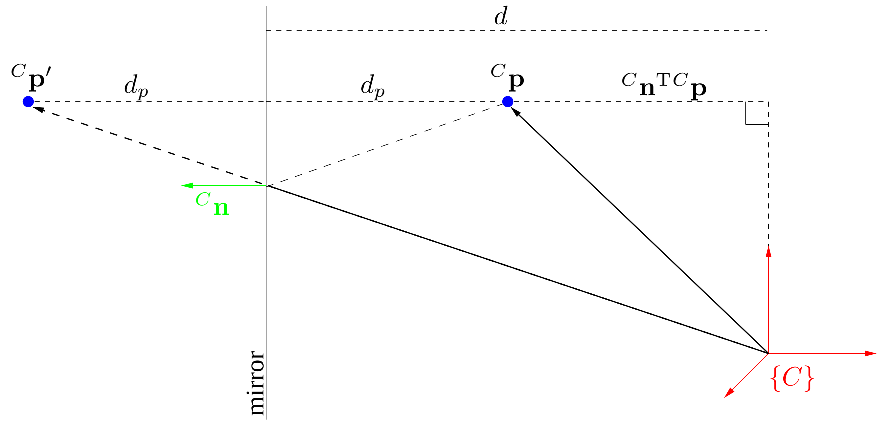

Figure 1: An observation of a point $^C\mathbf{p}'$ which is a reflection of $^C\mathbf{p}$. Only the reflected point is in the camera's field of view, the real point is not observed directly by the camera.

determine the 6-dof transformation $\{^C_B\mathbf{R},\ ^C\mathbf{p}_B\}$, between the camera and base frames of reference. The variable i is utilized throughout this technical report to index points. Regardless of solution method, this problem has up to four pairs of solutions, where for each pair, there is one solution lying in front of the center of perspectivity and one solution lying behind it [1].

Equation (6) differs from (4) in that the former contains a homogeneous transformation, while the latter contains both a homogeneous transformation and a reflection transformation. Our scenario is equivalent to a P3P in which an "imaginary" camera with a left-handed reference frame lies behind the mirror and observes the true points (not the reflections). To bring (4) into the same form as (6), we convert the imaginary camera to a right-handed system by premultiplying by a reflection about the y axis (although any axis can be chosen):

$$\begin{aligned}\begin{bmatrix}^{\breve{C}}\mathbf{p}'\\1\end{bmatrix} &= \begin{bmatrix}\left(\mathbf{I}_3 - 2\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}\right) & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}\left(\mathbf{I}_3 - 2^C\mathbf{n}^C\mathbf{n}^{\mathrm{T}}\right) & 2d^C\mathbf{n}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^C_B\mathbf{R} & ^C\mathbf{p}_B\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^B\mathbf{p}\\1\end{bmatrix}\\ &= \begin{bmatrix}^{\breve{C}}_B\mathbf{R} & ^{\breve{C}}\mathbf{p}_B\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^B\mathbf{p}\\1\end{bmatrix}\end{aligned} \tag{8}$$

where $\mathbf{e}_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^{\mathrm{T}}$. The origin of $\{\breve{C}\}$ coincides with that of $\{C\}$, their x and z axes are common, and their y axes lie in opposite directions. This modified problem is solved by any method which solves P3P, and we obtain $\{^{\breve{C}}_B\mathbf{R},\ ^{\breve{C}}\mathbf{p}_B\}$ which satisfies (8). As previously stated, there may be up to 4 such solutions, however, we are momentarily ignoring the multiple solutions for the clarity of discussion. The solution is reflected back to obtain $\{\mathbf{A},\ \mathbf{b}\}$:

$$\begin{aligned}\begin{bmatrix}^{C}\mathbf{p}'\\1\end{bmatrix} &= \begin{bmatrix}\left(\mathbf{I}_3 - 2\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}\right) & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^{\breve{C}}_B\mathbf{R} & ^{\breve{C}}\mathbf{p}_B\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^B\mathbf{p}\\1\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{A} & \mathbf{b}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}^B\mathbf{p}\\1\end{bmatrix}\end{aligned} \tag{9}$$

The transformation $\{\mathbf{A}, \mathbf{b}\}$, includes both a reflection and a rotation. Equating (4) and (9), we observe that:

$$\mathbf{A} = \left(\mathbf{I}_3 - 2^C\mathbf{n}^C\mathbf{n}^{\mathrm{T}}\right)\,^C_B\mathbf{R} \tag{10}$$

$$\mathbf{b} = \left(\mathbf{I}_3 - 2^C\mathbf{n}^C\mathbf{n}^{\mathrm{T}}\right)\,^C\mathbf{p}_B + 2d^C\mathbf{n} \tag{11}$$

Equations (10) and (11) relate the unknowns, $\{^C_B\mathbf{R}, ^C\mathbf{p}_B, ^C\mathbf{n}, d\}$, to the solution of the equivalent P3P, and it is this relationship which we exploit in our algorithm.

4.2 Solution from 3 points in 3 images

When only 2 points are observed, regardless of the number of images there is not enough information to determine the unknowns since 3 points are required to define the base frame. In the case of 2 points, or 3 or more collinear points, rotations about the line that the points lie on will not be observable. We conclude that at least 3 noncollinear points are required. From 1 image with 3 points there are not enough constraints to determine the unknowns (cf. (5)). As shown in Section 6, from 2 images with 3 points the number of constraints equals the number of unknowns. Unfortunately, rotations of the two mirror configurations about the axis of intersection between the two mirror planes are unobservable, and thus two images are not sufficient (cf. Section 6).

From 3 images with 3 points in each, there are 18 scalar measurements (cf. (5)) and 15 unknowns; 6 from $\{{}^C_B\mathbf{R}, {}^C\mathbf{p}_B\}$, and 3 for each mirror configuration $\{\mathbf{n}_j, d_j\}$ for $j = 1 \dots 3$ (cf. (4)). The variable j is utilized throughout this paper to index images (or equivalently, mirror configurations). This is an overdetermined system which is nonlinear in the unknown variables. Using P3P as an intermediate step, and momentarily ignoring multiple solutions, we have constraints of the form (10), (11) for each image.

We define vectors $\mathbf{m}_{jj'}$ for $j, j' \in \{1 \dots 3\}$, such that $\mathbf{m}_{jj'} \perp \mathbf{n}_j,\ \mathbf{n}_{j'}$, and note:

$$\begin{aligned}\mathbf{A}_j\mathbf{A}_{j'}^{\mathrm{T}}\mathbf{m}_{jj'} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_j{}^C\mathbf{n}_j^{\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_{j'}{}^C\mathbf{n}_{j'}^{\mathrm{T}}\right)\mathbf{m}_{jj'} \\ &= \mathbf{m}_{jj'}\end{aligned} \tag{12}$$

Thus, by solving for the eigenvector corresponding to the unit eigenvalue of $\mathbf{A}_j\mathbf{A}_{j'}^{\mathrm{T}}$, we determine $\mathbf{m}_{jj'}$ up to sign. These vectors allow the mirror normal vectors to be computed from cross-products as[2]

$$\mathbf{n}_1 = \mathbf{m}_{13} \times \mathbf{m}_{12}, \quad \mathbf{n}_2 = \mathbf{m}_{21} \times \mathbf{m}_{23}, \quad \mathbf{n}_3 = \mathbf{m}_{13} \times \mathbf{m}_{23} \tag{13}$$

The rotation matrix ${}^C_B\mathbf{R}$ can be computed independently from 3 sets of equations:

$${}^C_B\mathbf{R}_j = \left(\mathbf{I} - 2\mathbf{n}_j\mathbf{n}_j^{\mathrm{T}}\right)\mathbf{A}_j, \quad \text{for } j = 1 \dots 3 \tag{14}$$

In order to utilize all the available information, and to minimize numerical errors, we want to compute an "average" ${}^C_B\widehat{\mathbf{R}}$ from these 3 sets of equations. But summing them and dividing by 3 is inappropriate since the elements of a rotation matrix are nonlinear in the attitude configuration, and the property of orthonormality is not maintained. We mitigate this issue with the procedure described in Appendix A.

Once the rotation ${}^C_B\widehat{\mathbf{R}}$ and the mirror normal vectors $\mathbf{n}_j$ are determined, the remaining unknowns $\{{}^C\mathbf{p}_B,\ d_1,\ d_2,\ d_3\}$ appear linearly in the constraint equations (cf. (11))

$$\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} = \begin{bmatrix}\left(\mathbf{I} - 2\mathbf{n}_1\mathbf{n}_1^{\mathrm{T}}\right){}^C\mathbf{p}_B + 2d_1\mathbf{n}_1\\ \left(\mathbf{I} - 2\mathbf{n}_2\mathbf{n}_2^{\mathrm{T}}\right){}^C\mathbf{p}_B + 2d_2\mathbf{n}_2\\ \left(\mathbf{I} - 2\mathbf{n}_3\mathbf{n}_3^{\mathrm{T}}\right){}^C\mathbf{p}_B + 2d_3\mathbf{n}_3\end{bmatrix} = \begin{bmatrix}\left(\mathbf{I} - 2\mathbf{n}_1\mathbf{n}_1^{\mathrm{T}}\right) & 2\mathbf{n}_1 & \mathbf{0} & \mathbf{0}\\ \left(\mathbf{I} - 2\mathbf{n}_2\mathbf{n}_2^{\mathrm{T}}\right) & \mathbf{0} & 2\mathbf{n}_2 & \mathbf{0}\\ \left(\mathbf{I} - 2\mathbf{n}_3\mathbf{n}_3^{\mathrm{T}}\right) & \mathbf{0} & \mathbf{0} & 2\mathbf{n}_3\end{bmatrix}\begin{bmatrix}{}^C\mathbf{p}_B\\ d_1\\ d_2\\ d_3\end{bmatrix}$$

which is of the general form $\mathbf{D}\mathbf{x} = \mathbf{c}$ (where $\mathbf{D}$ is a 9×6 known matrix, $\mathbf{c}$ is a 9×1 known vector, and $\mathbf{x}$ is the 6×1 vector of unknowns). The least-squares solution for $\mathbf{x}$ in this linear system is $\mathbf{x} = \mathbf{D}^{\dagger}\mathbf{c}$, where $\mathbf{D}^{\dagger}$ denotes the Moore-Penrose generalized inverse of $\mathbf{D}$. Hence, $\{{}^C\mathbf{p}_B,\ d_1,\ d_2,\ d_3\}$ are computed as:

$$\begin{bmatrix}{}^C\mathbf{p}_B\\ d_1\\ d_2\\ d_3\end{bmatrix} = \begin{bmatrix}\left(\mathbf{I} - 2\mathbf{n}_1\mathbf{n}_1^{\mathrm{T}}\right) & 2\mathbf{n}_1 & \mathbf{0} & \mathbf{0}\\ \left(\mathbf{I} - 2\mathbf{n}_2\mathbf{n}_2^{\mathrm{T}}\right) & \mathbf{0} & 2\mathbf{n}_2 & \mathbf{0}\\ \left(\mathbf{I} - 2\mathbf{n}_3\mathbf{n}_3^{\mathrm{T}}\right) & \mathbf{0} & \mathbf{0} & 2\mathbf{n}_3\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} \tag{15}$$

From (13), (14), (54) and (15) the mirror configurations as well as the camera-to-base transformation are computed.

Up to now we assumed that the P3P solution was unique for each image, however, in general there may be up to 4 solutions per image. For each triplet of solutions, we obtain a transformation and set of mirror configurations as discussed. There are up to 64 such solutions arising from the $4 \times 4 \times 4$ possible combinations of P3P solutions. In the presence of pixel noise, none of the solutions will satisfy the measurements perfectly, hence, we choose the solution with the minimum reprojection error. Since there may be incorrect solutions resulting from degenerate measurement

[2] For the remainder of the paper, we drop the superscript 'C' from $\mathbf{n}_j,\ j = 1 \dots 3$.

sets (e.g., 3 images taken all from the same viewing angle), we need to remove outliers from the solution set. Using the method presented in [2], we determine the largest spectral cluster of solutions. In our implementation, we adopt the unit-quaternion representation of rotation, ${}^C\bar{q}_B$, which represents the rotation between frames $\{B\}$ and $\{C\}$ (cf. [3]). We define an affinity matrix, $\mathbf{L}$, in which each element is the Mahalanobis distance between a pair of solutions, indexed by μ and μ':

$$\mathbf{L}_{\mu\mu'} = \begin{bmatrix}\delta\boldsymbol{\theta}_{\mu\mu'}^{\mathrm{T}} & \delta\mathbf{p}_{\mu\mu'}^{\mathrm{T}}\end{bmatrix} \left[\left(\mathbf{H}_{\mu}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{H}_{\mu}\right)^{-1} + \left(\mathbf{H}_{\mu'}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{H}_{\mu'}\right)^{-1}\right]^{-1} \begin{bmatrix}\delta\boldsymbol{\theta}_{\mu\mu'} \\ \delta\mathbf{p}_{\mu\mu'}\end{bmatrix} \tag{16}$$

where $\delta\boldsymbol{\theta}_{\mu\mu'} = \delta\theta_{\mu\mu'}\mathbf{k}_{\mu\mu'}$ is the quaternion error-angle vector between ${}^C\bar{q}_B^{(\mu)}$ and ${}^C\bar{q}_B^{(\mu')}$, and $\delta\mathbf{p}_{\mu\mu'} = {}^C\mathbf{p}_B^{(\mu)} - {}^C\mathbf{p}_B^{(\mu')}$ is the difference between the translations. The matrices $\mathbf{H}_\mu$ and $\mathbf{H}_{\mu'}$ are the measurement Jacobians with respect to the transformation, and $\boldsymbol{\Gamma} = \sigma_{\boldsymbol{\eta}}^2\mathbf{I}$ is the covariance of the pixel noise (cf. Appendix B). We do not discuss spectral clustering further here, but refer the reader to [2] for details. Given N_s solutions which belong to the largest spectral cluster, we compute the transformation $\{{}^C_B\widehat{\mathbf{R}}, {}^C\widehat{\mathbf{p}}_B\}$. The rotation, ${}^C_B\widehat{\mathbf{R}}$, is determined using (54) for all quaternions in the cluster, and the translation, ${}^C\widehat{\mathbf{p}}_B$, is computed as the arithmetic mean of all the translations in the cluster.

5 Maximum Likelihood Estimation of the Transformation

We now proceed with the description of a maximum likelihood estimator (MLE) for determining the unknown transformation between the camera and base frames. We consider the case where N_p points in the base frame, denoted as ${}^B\mathbf{p}_i$, $i = 1 \ldots N_p$, are observed in N_c images of the camera. The observation of the ith point in the jth image ($j = 1 \ldots N_c$) is given by the equation (cf. (5)):

$$\mathbf{z}_{ij} = \mathbf{h}({}^{C_j}\mathbf{p}_i') + \boldsymbol{\eta}_{ij}, \qquad \text{where}$$
$${}^{C_j}\mathbf{p}_i' = \left(\mathbf{I}_3 - 2\frac{\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j}\right){}^C_B\mathbf{R}\,{}^B\mathbf{p}_i + \left(\mathbf{I}_3 - 2\frac{\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j}\right){}^C\mathbf{p}_B + 2\mathbf{v}_j$$

The vector, $\mathbf{v}_j = d_j\,{}^C\mathbf{n}_j$, describes the mirror configuration when the jth image is recorded. In the following, we use $\mathcal{Z}$ to denote the set of all available measurements.

It is interesting to examine the number of unknown parameters that exist in our problem. The unknown camera-to-base transformation $\{{}^C_B\mathbf{R}, {}^C\mathbf{p}_B\}$ introduces 6-dof, while each new image introduces an additional three unknowns, corresponding to the mirror configuration $\mathbf{v}_j$. Thus, the total number of unknown parameters is $6 + 3N_c$. On the other hand, each point observation provides two independent scalar measurements (the image coordinates of the projection), and thus we obtain a total of $2N_pN_c$ measurements, which we employ for estimating all unknown parameters. When the number of measurements equals or exceeds the number of unknowns, we expect to be able to compute a solution for all unknown parameters. It should be noted that a minimum of $N_p = 3$ noncollinear points observed in $N_c = 3$ images from different viewing angles are required for obtaining a solution. The 3 noncollinear points are necessary to define a coordinate frame. If only points on a single line are used, the frame's rotation about the line cannot be determined. Moreover, using only 2 camera views (or more which differ by rotations about a single axis) is not sufficient to uniquely determine the transformation between the camera and base frames (cf. Section 6).

Let the vector of all unknown parameters be denoted by $\mathbf{x}$. This vector comprises the unknown transformation, as well as the vectors $\mathbf{v}_j$, $j = 1 \ldots N_c$, that describe the mirror configuration. Thus $\mathbf{x}$ is:

$$\mathbf{x} = \begin{bmatrix}{}^C\mathbf{p}_B^{\mathrm{T}} & {}^C\bar{q}_B^{\mathrm{T}} & \mathbf{v}_1^{\mathrm{T}} & \ldots & \mathbf{v}_{N_c}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \tag{17}$$

where ${}^C\bar{q}_B$ is the unit quaternion representation of the rotation between frames $\{B\}$ and $\{C\}$. The likelihood of the measurements is given by:

$$\begin{aligned} L(\mathcal{Z};\mathbf{x}) &= \prod_{i=1}^{N_p}\prod_{j=1}^{N_c} p(\mathbf{z}_{ij};\mathbf{x}) \\ &= \prod_{i=1}^{N_p}\prod_{j=1}^{N_c} \frac{1}{2\pi\sigma_\eta^2}\exp\left[-\frac{\left(\mathbf{z}_{ij}-\mathbf{h}\left({}^{C_j}\mathbf{p}_i'\right)\right)^{\mathrm{T}}\left(\mathbf{z}_{ij}-\mathbf{h}\left({}^{C_j}\mathbf{p}_i'\right)\right)}{2\sigma_\eta^2}\right] \\ &= \prod_{i=1}^{N_p}\prod_{j=1}^{N_c} \frac{1}{2\pi\sigma_\eta^2}\exp\left[-\frac{(\mathbf{z}_{ij}-\mathbf{h}_{ij}(\mathbf{x}))^{\mathrm{T}}(\mathbf{z}_{ij}-\mathbf{h}_{ij}(\mathbf{x}))}{2\sigma_\eta^2}\right] \end{aligned}$$

where the dependence on $\mathbf{x}$ is explicitly shown (cf. (5)). Maximizing the likelihood is equivalent to maximizing its logarithm, which in turn is equivalent to minimizing the quantity:

$$J(\mathbf{x}) = \sum_{i=1}^{N_p}\sum_{j=1}^{N_c}(\mathbf{z}_{ij}-\mathbf{h}_{ij}(\mathbf{x}))^{\mathrm{T}}(\mathbf{z}_{ij}-\mathbf{h}_{ij}(\mathbf{x})) \tag{18}$$

The minimization of this cost function is a nonlinear least-squares problem, and thus we employ the Gauss-Newton iterative minimization algorithm for estimating $\mathbf{x}$. During each iteration k of the algorithm, the estimate is changed by:

$$\delta\mathbf{x}^{(k)} = \left(\sum_{i,j}\mathbf{H}_{ij}^{(k)\mathrm{T}}\mathbf{H}_{ij}^{(k)}\right)^{-1}\left(\sum_{i,j}\mathbf{H}_{ij}^{(k)\mathrm{T}}\left(\mathbf{z}_{ij}-\mathbf{h}_{ij}(\mathbf{x}^{(k)})\right)\right) \tag{19}$$

The Jacobian of the measurement $\mathbf{z}_{ij}$ with respect to $\mathbf{x}$ evaluated at the current iterate, $\mathbf{x}^{(k)}$, is given by:

$$\mathbf{H}_{ij} = \mathbf{H}_{c_{ij}}\begin{bmatrix}\mathbf{H}_{p_{ij}} & \mathbf{H}_{q_{ij}} & 0 & \dots & \underbrace{\mathbf{H}_{v_{ij}}}_{j-\text{th image}} & \dots & 0\end{bmatrix} \tag{20}$$

where $\mathbf{H}_{c_{ij}}$ is the Jacobian of the perspective projection model with respect to ${}^{C_j}\mathbf{p}_i'$:

$$\mathbf{H}_{c_{ij}} = \frac{1}{p_3}\begin{bmatrix}1 & 0 & -\frac{p_1}{p_3}\\ 0 & 1 & -\frac{p_2}{p_3}\end{bmatrix}$$

and $\mathbf{H}_{p_{ij}}$, $\mathbf{H}_{q_{ij}}$, and $\mathbf{H}_{v_{ij}}$, are the Jacobians of ${}^{C_j}\mathbf{p}_i'$ with respect to the position, rotation, and mirror configuration, respectively:

$$\begin{aligned} \mathbf{H}_{p_{ij}} &= \mathbf{I}_3 - 2\frac{\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j} \\ \mathbf{H}_{q_{ij}} &= \left(\mathbf{I}_3 - 2\frac{\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j}\right)\lfloor {}^C_B\mathbf{R}\,{}^R\mathbf{p}_i\times\rfloor \\ \mathbf{H}_{v_{ij}} &= 2\left(1-\frac{\mathbf{v}_j^{\mathrm{T}}\,{}^C\mathbf{p}_i}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j}\right)\mathbf{I}_3 - 2\frac{\mathbf{v}_j\,{}^C\mathbf{p}_i^{\mathrm{T}}}{\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j} + 4\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}\frac{\mathbf{v}_j^{\mathrm{T}}\,{}^C\mathbf{p}_i}{\left(\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_j\right)^2} \end{aligned}$$

It is worth mentioning that because this structure is sparse, the matrix to be inverted in the above equation is a sparse one. Thus, $\delta\mathbf{x}^{(k)}$ can be evaluated very efficiently (the computational cost of the operation can be shown to be linear in the number of images).

The parameter correction, $\delta\mathbf{x}^{(k)}$, has the following structure:

$$\delta\mathbf{x}^{(k)} = \begin{bmatrix}\delta{}^C\mathbf{p}_B^{(k)}\\ \delta\boldsymbol{\theta}^{(k)}\\ \delta\mathbf{v}_1^{(k)}\\ \vdots\\ \delta\mathbf{v}_{N_c}^{(k)}\end{bmatrix} \tag{21}$$

where all vectors on the right-hand side are 3×1 vectors. With this notation, the updates for the iterates of the parameters ${}^C\mathbf{p}_B$ and $\mathbf{v}_j$ are written as:

$$
\begin{aligned}
{}^C\mathbf{p}_B^{(k+1)} &= {}^C\mathbf{p}_B^{(k)} + \delta^C\mathbf{p}_B^{(k)} \\
\mathbf{v}_j^{(k+1)} &= \mathbf{v}_j^{(k)} + \delta\mathbf{v}_j^{(k)}, \quad j = 1 \dots N_c
\end{aligned}
$$

To ensure that the unit-length quaternion constraint is properly accounted for, a multiplicative error model is used for the quaternion iterates [3]:

$$
\begin{aligned}
{}^C\bar{q}_B^{(k+1)} &= \delta\bar{q}^{(k)} \otimes {}^C\bar{q}_B^{(k)}, \text{ with} \\
\delta\bar{q}^{(k)} &= \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}^{(k)} \\ \sqrt{1 - \frac{1}{4}\delta\boldsymbol{\theta}^{(k)\mathrm{T}}\delta\boldsymbol{\theta}^{(k)}} \end{bmatrix}.
\end{aligned} \tag{22}
$$

where $\otimes$ denotes quaternion multiplication. Employing this formulation for the quaternion updates enables us to have minimal error parametrization, since $\delta\boldsymbol{\theta}^{(k)}$ is a 3×1 vector.

After the Gauss-Newton algorithm converges to a minimum (convergence is determined by a threshold on the norm of $\delta\mathbf{x}^{(k)}$), the covariance of the resulting parameter estimates can be determined by the expression:

$$
\mathbf{P} = \sigma_\eta^2 \left(\sum_{i,j} \mathbf{H}_{ij}^{(k)\mathrm{T}} \mathbf{H}_{ij}^{(k)} \right)^{-1} \tag{23}
$$

6 Observability Study

Recall that the measurement model for a point reflected in a planar mirror and observed in a perspective projection model camera is given by:

$$
{}^C\mathbf{p}' = \left(\mathbf{I}_3 - 2{}^C\mathbf{n}{}^C\mathbf{n}^\mathrm{T}\right)\left({}^C_B\mathbf{R}{}^B\mathbf{p} + {}^C\mathbf{p}_B\right) + 2d{}^C\mathbf{n} \tag{24}
$$

$$
\mathbf{z} = \frac{1}{p_3}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \boldsymbol{\eta} = \mathbf{h}\left({}^C\mathbf{p}\right) + \boldsymbol{\eta}, \quad \text{where } {}^C\mathbf{p}' = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{25}
$$

In order to prove that a system is unobservable, it suffices to show that there exist at least two different states of the system which produce identical measurements. We show that for the case of two images (taken from different viewing angles), the measurements do not provide sufficient information to determine the state of the system uniquely, hence, the system is unobservable. The proof follows these steps:

1. Select a state $\{{}^C_B\mathbf{R}^{(1)}, {}^C\mathbf{p}_B^{(1)}, d_1^{(1)}, d_2^{(1)}, {}^C\mathbf{n}_1^{(1)}, {}^C\mathbf{n}_2^{(1)}\}$, which generates two observations, $\mathbf{z}_j^{(1)}$ for $j = 1, 2$, of point ${}^B\mathbf{p}$ via (24) and (25)[3].
2. Select a different state $\{{}^C_B\mathbf{R}^{(2)}, {}^C\mathbf{p}_B^{(2)}, d_1^{(2)}, d_2^{(2)}, {}^C\mathbf{n}_1^{(2)}, {}^C\mathbf{n}_2^{(2)}\}$ which generates a pair of measurements $\mathbf{z}_j^{(2)}$ of ${}^B\mathbf{p}$, for $j = 1, 2$.
3. Show that $\mathbf{z}_j^{(1)} = \mathbf{z}_j^{(2)}$ holds generically, for $j = 1, 2$.

6.1 Step 1

From the first state, we have measurements of the form

$$
\begin{aligned}
{}^C\mathbf{p}'_j &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_j^{(1)}{}^C\mathbf{n}_j^{(1)\mathrm{T}}\right)\left({}^C_B\mathbf{R}^{(1)}{}^B\mathbf{p} + {}^C\mathbf{p}_B^{(1)}\right) + 2d_j^{(1)}{}^C\mathbf{n}_j^{(1)} \\
&= \mathbf{A}_j^{(1)}{}^B\mathbf{p} + \mathbf{b}_j^{(1)}, \quad \text{for } j = 1, 2
\end{aligned} \tag{26}
$$

[3]The variable j indexes the images (or equivalently the mirror configurations)

The matrix $\mathbf{A}_j^{(1)}$ and the vector $\mathbf{b}_j^{(1)}$ are comprised of the state elements:

$$\mathbf{A}_j^{(1)} = \left(\mathbf{I}_3 - 2^C\mathbf{n}_j^{(1)C}\mathbf{n}_j^{(1)\mathrm{T}}\right) {}_B^C\mathbf{R}^{(1)} \tag{27}$$

$$\mathbf{b}_j^{(1)} = \left(\mathbf{I}_3 - 2^C\mathbf{n}_j^{(1)C}\mathbf{n}_j^{(1)\mathrm{T}}\right) {}^C\mathbf{p}_B^{(1)} + 2d_j^{(1)C}\mathbf{n}_j^{(1)} \tag{28}$$

Using (26) and (25), we have defined the state $\{{}_B^C\mathbf{R}^{(1)}, {}^C\mathbf{p}_B^{(1)}, d_1^{(1)}, d_2^{(1)}, {}^C\mathbf{n}_1^{(1)}, {}^C\mathbf{n}_2^{(1)}\}$, and the observations $\mathbf{z}_j^{(1)}$ for $j = 1, 2$, which the state generates.

6.2 Step 2

We construct a new state, which is different from the state in Step 1:

$${}^C\mathbf{n}_j^{(2)} = \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\, {}^C\mathbf{n}_j^{(1)}, \quad \text{for } j = 1, 2 \tag{29}$$

$$\begin{bmatrix} d_1^{(2)} \\ d_2^{(2)} \end{bmatrix} = \mathbf{G}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\, {}^C\mathbf{n}_1^{(1)} \tag{30}$$

$${}_B^C\mathbf{R}^{(2)} = \left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right) \mathbf{A}_1^{(1)} \tag{31}$$

$${}^C\mathbf{p}_B^{(2)} = \left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right) \mathbf{b}_1^{(1)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)} \tag{32}$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are an orthonormal basis of the plane which ${}^C\mathbf{n}_1^{(1)}$ and ${}^C\mathbf{n}_2^{(1)}$ span, such that ${}^C\mathbf{n}_j^{(1)} = c(\theta_j)\,\mathbf{u} + s(\theta_j)\,\mathbf{v}$ for $j = 1, 2$. The rotation matrix $\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)$, has axis of rotation $\mathbf{u}\times\mathbf{v}$, and angle of rotation α. The matrix $\mathbf{G}$ is defined as:

$$\mathbf{G} = \frac{1}{2s(2\delta\theta)} \begin{bmatrix} c(\delta\theta)\,\mathbf{r}^\mathrm{T}\mathbf{u} + s(\delta\theta)\,\mathbf{r}^\mathrm{T}\mathbf{v} & s(\delta\theta)\,\mathbf{r}^\mathrm{T}\mathbf{v} - c(\delta\theta)\,\mathbf{r}^\mathrm{T}\mathbf{u} \\ \mathbf{r}^\mathrm{T}\mathbf{v} & -\mathbf{r}^\mathrm{T}\mathbf{u} \end{bmatrix} \begin{bmatrix} \mathbf{u}^\mathrm{T} \\ \mathbf{v}^\mathrm{T} \end{bmatrix} \tag{33}$$

$$\text{where } \mathbf{r}^\mathrm{T}\mathbf{u} = 2d_1^{(1)}c(\theta_1) + 2d_2^{(1)}c(\theta_2) - 4d_2^{(1)}c(\delta\theta)\,c(\theta_1)$$

$$\text{and } \mathbf{r}^\mathrm{T}\mathbf{v} = 2d_1^{(1)}s(\theta_1) + 2d_2^{(1)}s(\theta_2) - 4d_2^{(1)}c(\delta\theta)\,s(\theta 1)$$

Here θ_1 is the angle from $\mathbf{u}$ to ${}^C\mathbf{n}_1^{(1)}$, θ_2 is the angle from $\mathbf{u}$ to ${}^C\mathbf{n}_2^{(1)}$, and $\delta\theta$ is the angle from ${}^C\mathbf{n}_1^{(1)}$ to ${}^C\mathbf{n}_2^{(1)}$. Using this new state, we define:

$$\begin{aligned} \mathbf{A}_j^{(2)} &= \left(\mathbf{I}_3 - 2^C\mathbf{n}_j^{(2)C}\mathbf{n}_j^{(2)\mathrm{T}}\right) {}_B^C\mathbf{R}^{(2)} \\ \mathbf{b}_j^{(2)} &= \left(\mathbf{I}_3 - 2^C\mathbf{n}_j^{(2)C}\mathbf{n}_j^{(2)\mathrm{T}}\right) {}^C\mathbf{p}_B^{(2)} + 2d_j^{(2)C}\mathbf{n}_j^{(2)}, \quad \text{for } j = 1, 2 \end{aligned} \tag{34}$$

State 2 generates a new pair of measurements $\{\mathbf{z}_1^{(2)}, \mathbf{z}_2^{(2)}\}$, of point ${}^B\mathbf{p}$, according to the measurement model.

6.3 Step 3

In order to show that $\mathbf{z}_j^{(1)} = \mathbf{z}_j^{(2)}$, for $j = 1, 2$, it suffices to show that ${}^C\mathbf{p}_j^{\prime(1)} = {}^C\mathbf{p}_j^{\prime(2)}$, which holds when $\mathbf{A}_j^{(1)} = \mathbf{A}_j^{(2)}$ and $\mathbf{b}_j^{(1)} = \mathbf{b}_j^{(2)}$. Thus, we will prove that $\mathbf{z}_j^{(1)} = \mathbf{z}_j^{(2)}$ by showing that $\mathbf{A}_j^{(1)} = \mathbf{A}_j^{(2)}$ and $\mathbf{b}_j^{(1)} = \mathbf{b}_j^{(2)}$.

6.3.1 Proof of $\mathbf{A}_1^{(2)} = \mathbf{A}_1^{(1)}$

From the definition

$$\begin{aligned} \mathbf{A}_1^{(2)} &= \left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right) {}_B^C\mathbf{R}^{(2)} \\ &= \left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right) \left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right) \mathbf{A}_1^{(1)} \end{aligned} \tag{35}$$

Since $\left(\mathbf{I}_3 - 2^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)$ is a Householder reflection across vector ${}^C\mathbf{n}_1^{(2)}$, and the product of a reflection with itself is the identity matrix:

$$\mathbf{A}_1^{(2)} = \mathbf{I}_3\mathbf{A}_1^{(1)} = \mathbf{A}_1^{(1)} \tag{36}$$

6.3.2 Proof of $\mathbf{b}_1^{(2)} = \mathbf{b}_1^{(1)}$

From the definition

$$\begin{aligned}
\mathbf{b}_1^{(2)} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right){}^C\mathbf{p}_B^{(2)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\left(\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{b}_1^{(1)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) + 2d_1^{(2)C}\mathbf{n}_1^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{b}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)} \\
&= \mathbf{I}_3\mathbf{b}_1^{(1)} - 2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)} \\
&= \mathbf{b}_1^{(1)}
\end{aligned} \tag{37}$$

6.3.3 Proof of $\mathbf{A}_2^{(2)} = \mathbf{A}_2^{(1)}$

From the definition

$$\begin{aligned}
\mathbf{A}_2^{(2)} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right){}_B^C\mathbf{R}^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{A}_1^{(1)} \\
&= \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right)\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}(\alpha)\,\mathbf{A}_1^{(1)}
\end{aligned} \tag{38}$$

The result of two reflections is equivalent to a rotation about the axis perpendicular to both reflection vectors, by an angle which is twice the angle between the vectors, i.e.,

$$\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right) = \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(2\delta\theta) \tag{39}$$

Recall that $\delta\theta$ is the angle from ${}^C\mathbf{n}_1^{(1)}$ to ${}^C\mathbf{n}_2^{(1)}$. Also, the vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to both ${}^C\mathbf{n}_1^{(1)}$ and ${}^C\mathbf{n}_2^{(1)}$, since $\mathbf{u}\times\mathbf{v}$ is normal to the plane in which they lie. We substitute (39) into (38), and exploit the property that rotations about the same axis commute:

$$\begin{aligned}
\mathbf{A}_2^{(2)} &= \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\,\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(2\delta\theta)\,\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}(\alpha)\,\mathbf{A}_1^{(1)} \\
&= \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(2\delta\theta)\,\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\,\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}(\alpha)\,\mathbf{A}_1^{(1)} \\
&= \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(2\delta\theta)\,\mathbf{I}_3\mathbf{A}_1^{(1)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right)\mathbf{A}_1^{(1)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right){}_B^C\mathbf{R}^{(1)}{}_B^C\mathbf{R}^{(1)\mathrm{T}}\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right)\mathbf{A}_1^{(1)} \\
&= \mathbf{A}_2^{(1)}\mathbf{A}_1^{(1)\mathrm{T}}\mathbf{A}_1^{(1)} \\
&= \mathbf{A}_2^{(1)}\mathbf{I}_3 = \mathbf{A}_2^{(1)}
\end{aligned} \tag{40}$$

6.3.4 Proof of $\mathbf{b}_2^{(2)} = \mathbf{b}_2^{(1)}$

From the definition

$$\begin{aligned}
\mathbf{b}_2^{(2)} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right){}^C\mathbf{p}_B^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{b}_1^{(1)} + 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{b}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right){}_B^C\mathbf{R}^{(2)}{}_B^C\mathbf{R}^{(2)\mathrm{T}}\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(2)C}\mathbf{n}_1^{(2)\mathrm{T}}\right)\mathbf{b}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \mathbf{A}_2^{(2)}\mathbf{A}_1^{(2)\mathrm{T}}\mathbf{b}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \mathbf{A}_2^{(1)}\mathbf{A}_1^{(1)\mathrm{T}}\mathbf{b}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)}
\end{aligned} \tag{41}$$

The first term in (41) is manipulated as follows:

$$\begin{aligned}
\mathbf{A}_2^{(1)}\mathbf{A}_1^{(1)\mathrm{T}}\mathbf{b}_1^{(1)} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right){}_B^C\mathbf{R}^{(1)}{}_B^C\mathbf{R}^{(1)\mathrm{T}}\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right)\left(\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_1^{(1)C}\mathbf{n}_1^{(1)\mathrm{T}}\right){}^C\mathbf{p}_B^{(1)} + 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right){}^C\mathbf{p}_B^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(-2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) \\
&= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right){}^C\mathbf{p}_B^{(1)} + 2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_2^{(1)C}\mathbf{n}_2^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(-2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) \\
&= \mathbf{b}_2^{(1)} - 2d_2^{(1)C}\mathbf{n}_2^{(1)} - \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)2d_1^{(1)C}\mathbf{n}_1^{(1)} \qquad (42)
\end{aligned}$$

Substituting (42) into (41):

$$\begin{aligned}
\mathbf{b}_2^{(2)} &= \mathbf{b}_2^{(1)} - 2d_2^{(1)C}\mathbf{n}_2^{(1)} - \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)2d_1^{(1)C}\mathbf{n}_1^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)2d_1^{(2)C}\mathbf{n}_1^{(2)} + 2d_2^{(2)C}\mathbf{n}_2^{(2)} \\
&= \mathbf{b}_2^{(1)} + \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) - \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(2d_2^{(2)C}\mathbf{n}_2^{(2)} - 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) \qquad (43)
\end{aligned}$$

The last term in (43) is manipulated as follows:

$$\begin{aligned}
\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(2d_2^{(2)C}\mathbf{n}_2^{(2)} - 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) &= 2\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(2)} & {}^C\mathbf{n}_2^{(2)}\end{bmatrix}\begin{bmatrix}d_1^{(2)} \\ d_2^{(2)}\end{bmatrix} \\
&= 2\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}(\alpha) \\
&\quad \cdot \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\begin{bmatrix}d_1^{(2)} \\ d_2^{(2)}\end{bmatrix} \\
&= 2\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\begin{bmatrix}d_1^{(2)} \\ d_2^{(2)}\end{bmatrix} \qquad (44)
\end{aligned}$$

Recalling the definition of $\begin{bmatrix}d_1^{(2)} & d_2^{(2)}\end{bmatrix}^{\mathrm{T}}$, we write:

$$\begin{aligned}
&= 2\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\mathbf{G}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha){}^C\mathbf{n}_1^{(1)} \\
&= 2\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\mathbf{M}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha){}^C\mathbf{n}_1^{(1)} \qquad (45)
\end{aligned}$$

Where we have implicitly defined $\mathbf{M}$ as the product of the terms between the rotation matrices. Expanding $\mathbf{M}$, we have:

$$\begin{aligned}
\mathbf{M} &= \left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\mathbf{G} \\
&= \frac{1}{2s(2\delta\theta)}\left(\mathbf{I}_3 - 2{}^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\begin{bmatrix}c(\delta\theta)\mathbf{r}^{\mathrm{T}}\mathbf{u} + s(\delta\theta)\mathbf{r}^{\mathrm{T}}\mathbf{v} & s(\delta\theta)\mathbf{r}^{\mathrm{T}}\mathbf{v} - c(\delta\theta)\mathbf{r}^{\mathrm{T}}\mathbf{u} \\ \mathbf{r}^{\mathrm{T}}\mathbf{v} & -\mathbf{r}^{\mathrm{T}}\mathbf{u}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}}\end{bmatrix} \\
&= a_1\mathbf{u}\mathbf{u}^{\mathrm{T}} + a_2\mathbf{u}\mathbf{v}^{\mathrm{T}} + a_2\mathbf{v}\mathbf{u}^{\mathrm{T}} - a_1\mathbf{v}\mathbf{v}^{\mathrm{T}} \qquad (46)
\end{aligned}$$

where
$$\begin{aligned}
a_1 &= 2c^2(\delta\theta)c(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{v} - c(\delta\theta)c(\theta_1)\mathbf{r}^{\mathrm{T}}\mathbf{v} + 2s(\delta\theta)c(\delta\theta)c(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{u} - s(\delta\theta)c(\theta_1)\mathbf{r}^{\mathrm{T}}\mathbf{u} - c(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{v} \\
a_2 &= 2c^2(\delta\theta)s(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{v} - c(\delta\theta)s(\theta_1)\mathbf{r}^{\mathrm{T}}\mathbf{v} + 2s(\delta\theta)c(\delta\theta)s(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{u} - s(\delta\theta)s(\theta_1)\mathbf{r}^{\mathrm{T}}\mathbf{u} - s(\theta_2)\mathbf{r}^{\mathrm{T}}\mathbf{v}
\end{aligned}$$

Employing the Rodriguez Formula, $\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha) = c(\alpha)\mathbf{I}_3 + s(\alpha)\lfloor(\mathbf{u}\times\mathbf{v})\ \times\rfloor + (1 - c(\alpha))(\mathbf{u}\times\mathbf{v})(\mathbf{u}\times\mathbf{v})^{\mathrm{T}}$:

$$\begin{aligned}
\mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\mathbf{M} &= a_1c(\alpha)\mathbf{u}\mathbf{u}^{\mathrm{T}} + a_1s(\alpha)\mathbf{v}\mathbf{u}^{\mathrm{T}} + a_2c(\alpha)\mathbf{u}\mathbf{v}^{\mathrm{T}} + a_2s(\alpha)\mathbf{v}\mathbf{v}^{\mathrm{T}} + a_2c(\alpha)\mathbf{v}\mathbf{u}^{\mathrm{T}} \\
&\quad - a_2s(\alpha)\mathbf{u}\mathbf{u}^{\mathrm{T}} - a_1c(\alpha)\mathbf{v}\mathbf{v}^{\mathrm{T}} + a_1s(\alpha)\mathbf{u}\mathbf{v}^{\mathrm{T}} \qquad (47) \\
\mathbf{M}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}(\alpha) &= a_1c(\alpha)\mathbf{u}\mathbf{u}^{\mathrm{T}} + a_1s(\alpha)\mathbf{u}\mathbf{v}^{\mathrm{T}} + a_2c(\alpha)\mathbf{u}\mathbf{v}^{\mathrm{T}} - a_2s(\alpha)\mathbf{u}\mathbf{u}^{\mathrm{T}} + a_2c(\alpha)\mathbf{v}\mathbf{u}^{\mathrm{T}} + a_2s(\alpha)\mathbf{v}\mathbf{v}^{\mathrm{T}} \\
&\quad - a_1c(\alpha)\mathbf{v}\mathbf{v}^{\mathrm{T}} + a_1s(\alpha)\mathbf{v}\mathbf{u}^{\mathrm{T}} \\
&= \mathbf{R}_{\mathbf{u}\times\mathbf{v}}(\alpha)\mathbf{M} \qquad (48)
\end{aligned}$$

Plugging (48) into equation (45) we have

$$
\begin{aligned}
\left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(2d_2^{(2)C}\mathbf{n}_2^{(2)} - 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) &= 2\mathbf{R}_{\mathbf{u}\times\mathbf{v}}\left(\alpha\right)\mathbf{M}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}\left(\alpha\right){}^C\mathbf{n}_1^{(1)} \\
&= 2\mathbf{M}\mathbf{R}_{\mathbf{u}\times\mathbf{v}}^{\mathrm{T}}\left(\alpha\right)\mathbf{R}_{\mathbf{u}\times\mathbf{v}}\left(\alpha\right){}^C\mathbf{n}_1^{(1)} \\
&= 2\mathbf{M}^C\mathbf{n}_1^{(1)} \\
&= 2\left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\mathbf{G}^C\mathbf{n}_1^{(1)} \\
&= 2\left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\begin{bmatrix}-{}^C\mathbf{n}_1^{(1)} & {}^C\mathbf{n}_2^{(1)}\end{bmatrix}\begin{bmatrix}d_1^{(1)} \\ d_2^{(1)}\end{bmatrix} \\
&= \left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) \qquad (49)
\end{aligned}
$$

Finally, plugging (49) into (43), we have

$$
\begin{aligned}
\mathbf{b}_2^{(2)} &= \mathbf{b}_2^{(1)} + \left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) - \left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(2)C}\mathbf{n}_2^{(2)\mathrm{T}}\right)\left(2d_2^{(2)C}\mathbf{n}_2^{(2)} - 2d_1^{(2)C}\mathbf{n}_1^{(2)}\right) \\
&= \mathbf{b}_2^{(1)} + \left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) - \left(\mathbf{I}_3 - 2^C\mathbf{n}_2^{(1)C}\mathbf{n}_2^{(1)\mathrm{T}}\right)\left(2d_2^{(1)C}\mathbf{n}_2^{(1)} - 2d_1^{(1)C}\mathbf{n}_1^{(1)}\right) \\
&= \mathbf{b}_2^{(1)} \qquad (50)
\end{aligned}
$$

6.4 Remarks

We have shown that when two images (mirror configurations) are available there are two different states of the system which generate identical measurements. Hence, the system is unobservable. We began by selecting state 1, and defining the resulting measurements. The second set of state parameters was generated by rotating the mirrors about their axis of intersection ($\mathbf{u} \times \mathbf{v}$ vector) by an angle α. This rotation did not affect the measurements, and hence we say that it is the unobservable mode in our system. It is interesting to note that when only two images are available, adding more points will not add new information.

Appendix A

We adopt the quaternion notation from [3] and denote the quaternion of rotation arising from the jth set of equations as $\bar{q}_j$, which corresponds to ${}^C_B\mathbf{R}_j$ (cf. (14)). Assuming that $\bar{q}$ is the true solution, and employing the small error-angle approximation, we write the following expression for the quaternion error:

$$\bar{q}_j \otimes \bar{q}^{-1} \simeq \begin{bmatrix}\mathbf{k}_j\delta\theta_j \\ 1\end{bmatrix} \quad \text{for } j = 1\ldots 3 \qquad (51)$$

where $\otimes$ denotes quaternion multiplication, $\mathbf{k}_j$ is the unit-vector axis of rotation, and $\delta\theta_j$ is the error angle between the two quaternions. Rewriting this last expression as a matrix-vector multiplication [3], yields,

$$\mathcal{L}\left(\bar{q}_j\right)\bar{q}^{-1} = \begin{bmatrix}\mathbf{k}_j\delta\theta_j \\ 1\end{bmatrix} \qquad (52)$$

where $\mathcal{L}\left(\bar{q}_j\right)$, is the left-side quaternion multiplication matrix parameterized by $\bar{q}_j$. Projecting this relation, we have

$$\mathbf{P}\mathcal{L}\left(\bar{q}_j\right)\bar{q}^{-1} = \mathbf{k}_j\delta\theta_j, \quad \text{where } \mathbf{P} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_{3\times 1}\end{bmatrix},\ j = 1\ldots 3 \qquad (53)$$

Stacking these relations, we see

$$\begin{bmatrix}\mathbf{P}\mathcal{L}\left(\bar{q}_1\right) \\ \mathbf{P}\mathcal{L}\left(\bar{q}_2\right) \\ \mathbf{P}\mathcal{L}\left(\bar{q}_3\right)\end{bmatrix}\bar{q}^{-1} = \begin{bmatrix}\mathbf{k}_1\delta\theta_1 \\ \mathbf{k}_2\delta\theta_2 \\ \mathbf{k}_3\delta\theta_3\end{bmatrix} \qquad (54)$$

Ideally, we would like to find an estimate $\hat{\bar{q}}^{-1}$ of $\bar{q}^{-1}$, such that the right-hand side is the zero vector. Hence, we choose $\hat{\bar{q}}^{-1}$ to minimize the 2-norm of the right-hand side. This occurs exactly when $\hat{\bar{q}}^{-1} = \mathbf{v}\left(\sigma_{min}\right)$, i.e., we select $\hat{\bar{q}}^{-1}$ to be the right singular vector corresponding to the minimum singular value of the 3×4 matrix multiplying $\bar{q}^{-1}$ in (54). After finding $\hat{\bar{q}}^{-1}$ by SVD, we compute the rotational matrix ${}^{C}_{B}\widehat{\mathbf{R}} = \mathbf{R}\left(\hat{\bar{q}}\right)$, which is the rotational matrix parameterized by quaternion $\hat{\bar{q}}$.

Appendix B

We now present the derivation for the Mahalanobis distance between two solutions $\mathbf{x}_\mu = \{{}^{C}\bar{q}_B^{(\mu)}, {}^{C}\mathbf{p}_B^{(\mu)}\}$, for $\mu = 1, 2$, which are computed analytically from different sets of measurements $\mathcal{Z}_\mu$. Since the measurement function is nonlinear (cf. (4) and (5)), the posterior pdf, $p\left(\mathbf{x}_\mu \mid \mathcal{Z}_\mu\right)$, is multi-modal. Assuming the true solution is $\mathbf{x}_t$, we approximate the posterior pdf by a gaussian, i.e.,

$$p\left(\mathbf{x}_\mu \mid \mathcal{Z}_\mu\right) \sim \mathcal{N}\left(\mathbf{x}_t, \left(\mathbf{H}_\mu^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{H}_\mu\right)^{-1}\right) \tag{55}$$

$$\text{where } \mathbf{H}_\mu = \nabla_{\mathbf{x}} h\left(\mathbf{x}\right)|_{\mathbf{x}=\mathbf{x}_\mu} \text{ and } \mathbf{\Gamma} = \sigma_\eta^2 \mathbf{I}$$

The matrix $\mathbf{H}_\mu$ is the Jacobian of the measurement with respect to the state (cf. (20)), and Γ is the covariance of the measurement noise. Since the solutions are obtained from different sets of image measurements, they are independent. Noting that

$$\begin{aligned}\tilde{\mathbf{x}} &= \mathbf{x}_1 - \mathbf{x}_2 \\ &= \left(\mathbf{x}_t + \tilde{\mathbf{x}}_1\right) - \left(\mathbf{x}_t + \tilde{\mathbf{x}}_2\right) \\ &= \tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\end{aligned} \tag{56}$$

We evaluate the mean

$$\begin{aligned}E\left(\tilde{\mathbf{x}}\right) &= E\left(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\right) \\ &= E\left(\tilde{\mathbf{x}}_1\right) - E\left(\tilde{\mathbf{x}}_2\right) = 0\end{aligned} \tag{57}$$

and the covariance

$$\begin{aligned}E\left(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}\right) &= E\left(\left(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\right)\left(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\right)^{\mathrm{T}}\right) \\ &= E\left(\tilde{\mathbf{x}}_1\tilde{\mathbf{x}}_1^{\mathrm{T}}\right) + E\left(\tilde{\mathbf{x}}_2\tilde{\mathbf{x}}_2^{\mathrm{T}}\right) \\ &= \left(\mathbf{H}_1^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{H}_1\right)^{-1} + \left(\mathbf{H}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{H}_2\right)^{-1}\end{aligned} \tag{58}$$

Now, we define the Mahalanobis distance

$$d_{1,2} = \tilde{\mathbf{x}}^{\mathrm{T}}\left(\left(\mathbf{H}_1^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{H}_1\right)^{-1} + \left(\mathbf{H}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{H}_2\right)^{-1}\right)\tilde{\mathbf{x}} \tag{59}$$

which is the quadratic distance between the solutions, weighted by their covariance.

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